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LETTER TO THE EDITOR

Miura-type transformations

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Abstract. In this letter Miura-type transformations are considered for generalized KdV equations of the form $u_y + u_{xxx} + f(u, u_x) = 0$. The nonlinear forms of this equation that admit such transformations are completely classified. For the same equations Bäcklund transformations are derived. Finally we investigate a wider class of transformations.

The Miura transformation [1] $u = u'_x - u'^2$ relates the Korteweg–de Vries (KdV) equation $u_y + u_{xxx} + 6uu_x = 0$ and the modified KdV equation $u'_y + u'_{xxx} - 6u'^2 u'_x = 0$. Miura's result is analogous to the Hopf–Cole transformation of Burgers' equation to the diffusion equation [2, 3], except that Miura's is a transformation between two nonlinear equations, neither of which can be solved in general. Nevertheless, this transformation is the key to prove the existence of an infinite number of conservation laws for the KdV equation [4]. It also provides the starting point for the inverse scattering method which provides exact solution for the same equation [4]. A generalization of the Hopf–Cole transformation is given by Sachdev [5].

We present a theorem which gives a class of Miura-type transformations for generalized KdV equations. Motivated by this theorem, theorem 2 is introduced to give the Bäcklund transformations (BTs) which relate the same equations. Theorem 2 can be seen as a generalization of Lamb's result [6], who, also motivated by the Miura transformation, derived BTs connecting the KdV and the modified KdV equations. Finally, in the spirit of the work of Sophocleous and Kingston [7], who derived transformations of the class $u = F(u', u'_x, u'_y)$ for equations of the form $u_{xy} = f(u, u_x)$, we explore the range of functions $f(u, u_x)$ and $f'(u', u'_x)$ for which the equations $u_y + u_{xxx} + f(u, u_x) = 0$ and $u'_y + u'_{xxx} + f'(u', u'_x) = 0$ admit transformations of the class $u = F(u', u'_x, u'_y, u'_{xx}, u'_{xy}, u'_{yy})$.

Throughout we shall use the standard notation p, q, r, s and t for u derivatives, similarly for u' derivatives and $\alpha' = u'_{xxx}$, $\beta' = u'_{xxy}$, $\gamma' = u'_{xyy}$ and $\delta' = u'_{yyy}$.

Theorem 1. The pair of partial differential equations

$$u_y + u_{xxx} + \frac{1}{\lambda^2}[\tfrac{3}{2}\mu(u - \mu\tau)^2 - 3\tau(1 - \mu^2)(u - \theta) - \tfrac{3}{2}(\phi^2 + \mu\theta^2) + \nu]\, u_x = 0 \tag{1}$$

$$u'_y + u'_{xxx} + \tfrac{1}{2}\mu u'^3_x - \frac{3}{2\lambda^2} c^2_{u'} u'_x + \frac{\nu}{\lambda^2} u'_x = 0 \tag{2}$$

are related by the transformation

$$u = \lambda u'_x + c \tag{3}$$

where c is a function of u' and is given by the ordinary differential equation

$$c_{u'u'}+\mu c=\tau \qquad c_{u'u'}\neq 0 \tag{4}$$

where λ, ν, μ, τ, ϕ and θ are all constants, $\mu=0$, -1, 1 and ϕ, θ are constants of integration obtained by solving equation (4). These constants may be fixed without loss of generality. In particular by appropriate scaling of u and u' and writing equations (1) and (2) in a moving frame we may take $\lambda=1$ and $\nu=0$. Nevertheless, we have not fixed these constants, so one can easily deduce special cases of the theorem 1.

Proof (outline). For $\mu=0$, -1, 1 we obtain from equations (3) and (4) that

$$u=\lambda p'+\frac{\tau}{2}u'^2+\phi u'+\theta \tag{5}$$

$$u=\lambda p'+\phi\sinh u'+\theta\cosh u'-\tau \tag{6}$$

$$u=\lambda p'+\phi\sin u'+\theta\cos u'+\tau \tag{7}$$

respectively. For each of the separate forms (5)-(7), u_x, u_y and u_{xxx} can be evaluated in terms of u' and its derivatives. Recalling that $\alpha'=-[q'+\frac{1}{2}\mu p'^3-3p'c_{u'}^2/(2\lambda^2)+\nu p'/\lambda^2]$, from equation (2), these expressions for u_y, u_x and u_{xxx} will satisfy equation (1).

We note that on setting $\mu=\phi=\theta=0$, $\lambda=1$, $\tau=-2$, we obtain the transformation $u=p'-u'^2$ which relates the KdV equation $q+\alpha+6up=0$ and the modified KdV equation $q'+\alpha'-6u'p'^2=0$. This is the well known Miura transformation. Also if we let $\mu=\theta=0$, $\nu=\frac{3}{2}$, $\tau=-2\lambda^2$, $\phi=1$ we find that the equations $q+\alpha+6up=0$ and $q'+\alpha'-6(\lambda^2u'^2-u')p'=0$ are related by $u=\lambda p'-\lambda^2u'^2+u'$. The latter is known as Gardner's transformation [4]. Also when $\mu=\pm1$ we obtain the well known relationship between the modified KdV equation to an equation that was first introduced by Calogero and Degasperis [8].

Let us write equation (3) in the form

$$p'=\frac{1}{\lambda}(u-c) \tag{8}$$

and also let

$$q'=\Phi(u',u,p,r). \tag{9}$$

Considering (8) and (9) and taking $\mu=0$, -1, 1, separately, we can derive BTs for equations (1) and (2) using the method of Clairin [9]. Without presenting the detailed computations we summarize the results in the following theorem.

Theorem 2. The BTs

$$p'=\frac{1}{\lambda}(u-c)$$

$$q'=-\frac{1}{\lambda}r+\frac{1}{\lambda^2}pc_{u'}-\tfrac{1}{2}\mu p'^3+\frac{1}{2\lambda^2}p'c_{u'}^2+\frac{1}{\lambda}p'^2c_{u'u'}-\frac{1}{\lambda^2}\nu p'$$

relate the partial differential equations (1) and (2), where c is given by equation (4). The constants are as defined in theorem 1.

The proof is straightforward. Theorem 2 is a generalization of the BTs obtained by Lamb [6], which relate the KdV and the modified KdV equations. Choosing $\mu=\nu=\phi=\theta=0$, $\lambda=1$ and $\tau=-2$ in the above theorem gives the result obtained by Lamb. Also setting $\mu=-1$, $\lambda^2=\frac{1}{4}$, $A=-\frac{3}{2}(\phi+\theta)^2$, $B=-\frac{3}{2}(\phi-\theta)^2$, $D=-3(\phi^2-\theta^2)$ and $\nu=0$ equation (2) becomes the Calogero and Degasperis equation

$$q'+\alpha'-\tfrac{1}{2}p'^3+[A\exp(2u')+B\exp(-2u')+D]p'=0$$

where, making the transformation $u'\to\frac{1}{2}u'$ this equation takes the same form as in [8]. The above equation is linked to the modified KdV equation

$$q+\alpha-6[(u+\tau)^2-\tfrac{1}{3}D]p=0$$

by the BTs

$$p'=\pm2(u-c)$$

$$q'=\mp2r'+4pc_{u'}+\tfrac{1}{2}p'^3+2p'c_{u'}^2\pm p'^2c_{u'u'}$$

where $c=\phi\sinh u'+\theta\cosh u'-\tau$.

We consider the two partial differential equations

$$q+\alpha+f(u,p)=0 \tag{10}$$

and

$$q'+\alpha'+f'(u',p')=0 \tag{11}$$

and the transformations of the form

$$u=F(u',p',q',r',s',t'). \tag{12}$$

In the subsequent analysis we shall determine the range of functions f and f' for which equations (10) and (11) admit transformations of the class (12). We shall exclude cases where both f and f' are linear in both of their arguments. Point transformations are also excluded.

Upon differentiating (12) with respect to x and y respectively, we obtain

$$p=p'F_{u'}+r'F_{p'}+s'F_{q'}+\alpha'F_{r'}+\beta'F_{s'}+\gamma'F_{t'} \tag{13}$$

$$q=q'F_{u'}+s'F_{p'}+t'F_{q'}+\beta'F_{r'}+\gamma'F_{s'}+\delta'F_{t'}. \tag{14}$$

Remembering that $\alpha'=-q'-f'$, from (11), and differentiating (13) twice with respect to x a similar expression for α is obtained. Upon substitution of this expression for α and (14) into equation (10) we obtain

$$q+\alpha+f=E(u',p',q',r',s',t',\beta',\gamma',\varepsilon')=0 \tag{15}$$

where $\varepsilon'=u'_{xxyy}$, for some function E which may be calculated explicitly in terms of F, its derivatives, f', its derivatives, u', p', q', r', s', t', β', γ' and ε'. In view of (10) E must be identically zero with u', p', q', r', s', t', β', γ' and ε' regarded as independent variables.

The following calculations in which the detailed computations have been omitted, were performed with the assistance of the algebraic manipulation package REDUCE [10]. The coefficient of ε' in (15) implies that $F=\lambda_5t'+F_1(u',p',q',r',s')$, F_1 being a function of the indicated arguments and λ_5 a constant. Upon differentiation (15) twice with respect to γ' we obtain $\lambda_5^2f_{pp}=0$. Taking $\lambda_5\neq0$ leads to excluded (linear) forms of f and f'. Hence, one needs to take $\lambda_5=0$. Now the coefficient of t' implies that

$F(=F_1)=\lambda_4 s'+F_2(u',p',q',r')$ and $E_{\beta'\beta'}=\lambda_4^2 f_{pp}=0$. This time we must take $\lambda_4=0$ for non-excluded cases. Similarly, picking the coefficient of β' and then differentiating E twice with respect to s' we deduce that F is also independent of q'. Hence,

$$u=F(u',p',r').$$

The coefficient of s' implies that $F=\lambda_2 r'+A(u',p')$, where A is a function of u' and p' and λ_2 is a constant. Now $E_{q'q'}=\lambda_2^2 f_{pp}=0$. Here $\lambda_2\neq 0$ does not lead to excluded cases. Therefore one needs to split the analysis into two disjoint cases: (i) $\lambda_2\neq 0$ and f is linear in p and (ii) $\lambda_2=0$.

(i) Calculations of $E_{q'}$, $E_{r'r'}$ and $E_{r'}$ lead to the following results

$$F=\lambda_2 r'-\tfrac{1}{6}\lambda_2^2\mu_1 p'^2+a(u')p'+b(u') \tag{16}$$

$$f=(\mu_1 u+\mu_2)p \tag{17}$$

$$f'=-\tfrac{1}{18}\lambda_2^2\mu_1^2 p'^3+\tfrac{1}{2}\mu_1 a p'^2+\mu_1 b p'+\mu_2 p'+\frac{3}{2\lambda_2}a_{u'}p'^2 \tag{18}$$

where

$$a=\nu_1\exp(-\tfrac{1}{3}\lambda_2\mu_1 u')+\nu_2$$

$$b=\nu_5\exp(\tfrac{2}{3}\lambda_2\mu_1 u')-\frac{3\nu_1^2}{8\mu_1\lambda_2^2}\exp(-\tfrac{2}{3}\lambda_2\mu_1 u')-\frac{\nu_1\nu_2}{\mu_1\lambda_2^2}\exp(-\tfrac{1}{3}\lambda_2\mu_1 u')-\frac{3(\nu_2^2+\nu_4)}{4\mu_1\lambda_2^2}$$

and where μ_1, μ_2, ν_1, ν_2, ν_3, ν_4 and ν_5 are all constants. The constant μ_1 has been taken as non-zero because otherwise both f and f' are linear.

Finally equating coefficients of powers of p' in $E=0$, we deduce that we must either have (a) $\nu_2=\nu_3=0$ or (b) $\nu_2\neq 0$, $\nu_1=\nu_5=0$. These two subcases give all possible forms of F, f and f' (equations (16)–(18)). As an example, if we set $\mu_1=6$ and $\mu_2=\nu_1=\nu_2=\nu_3=\nu_4=\nu_5=0$ we obtain the transformation $u=\lambda_2 r'-\lambda_2^2 p'^2$ which relates the KdV equation and the equation $q'+\alpha'-2\lambda_2^2 p'^3=0$.

(ii) Differentiating E firstly, with respect to q' and secondly, twice with respect to r' yields

$$F=\lambda_1 p'+c(u') \qquad f=L(u)p+M(u)$$

respectively, where L and M are functions of u, c is a function of u' and λ_1 is a constant. Because we exclude point transformations, $\lambda_1\neq 0$. Equating coefficients of powers of r' in $E=0$ now gives

$$\lambda_1 L=\lambda_1 f'_{p'}-3p'c_{u'u'} \tag{19}$$

$$p'c_{u'}L+M=f'c_{u'}-p'^3 c_{u'u'u'}+\lambda_1 p' f'_{u'}. \tag{20}$$

These two equations impose restrictions on the functions L, M, f' and c and that ultimately enables the transformations to be derived.

Without presenting any more detailed computations, we state the results of this case. If $c_{u'u'}\neq 0$, then equations (19) and (20) lead to theorem 1. If $c=\nu u'+\tau$, one finds the following two trivial transformations:

(a) $u=\lambda_1 p'+\nu u'+\tau$ relates the equations $q+\alpha+[\lambda_1 M'(u)+\mu]p+M(u)=0$ and $q'+\alpha'+[M(\lambda_1 p'+\nu u'+\tau)+\mu p']/\nu=0$, where M is an arbitrary function, M' is its derived function, μ, ν and τ are constants and ν is non-zero.

(b) $u=\lambda_1 p'+\tau$ relates the equations $q+\alpha+J'((u-\tau)/\lambda_1)p+\mu u-\tau\mu=0$ and $q'+\alpha'+\mu u'+J(p')=0$, where J is an arbitrary function, J' is its derived, μ and τ are constants.

If we let $\mu=\tau=0$ and $J=-2\lambda_1^2 p'^3$ in the second transformation, we find that the modified KdV equation and the equation $q'+\alpha'-2\lambda_1^2 p'^3=0$ are related by $u=\lambda_1 p'$. Combining this transformation and the Miura transformation we obtain the transformation which was given as example in case (i).

Therefore theorem 1 and the transformations (a) and (b) provide the complete set of Miura-type transformations for the equations (10) and (11). Using these two trivial transformations ((a) and (b)) we can also derive BTs for the same equations.

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